ETALÉ FUNDAMENTAL GROUP AND DESSIN D'ENFANTS

AAYUSH DUTTA

ABSTRACT. Given some ring R, we note that the topological properties of $\mathrm{Spec}(R)$ often correspond to algebraic properties of R. For example, disconnectedness of $\mathrm{Spec}(R)$ corresponds to the existence of non-trivial idempotents, R being Noetherian corresponds to $\mathrm{Spec}(R)$ being Noetherian, as a topological space. Another topological invariant we may choose to look at is $\pi_1(\mathrm{Spec}(R))$, the fundamental group. In this paper, we aim to explore how $\pi_1(\mathrm{Spec}(R))$ can be seen in terms of R (for certain classes of rings) by deriving $\pi_1^{et}(R)$ from R, the etalé fundamental group. We then aim to apply this idea to understand $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via the $\pi_1^{\text{ét}}\left(\mathrm{Spec}\,\mathbb{C}\left[x, \frac{1}{x(x-1)}\right]\right)$ and discuss what elements in this object look like.

CONTENTS

1. INTRODUCTION TO $\pi_1(X)$ AND COVERING SPACES: A TOPOLOGICAL PRECURSOR

The process of constructing topological invariants often rise from trying to distinguish between (seemingly) non-homeomorphic topological spaces. For example, the basic point-set

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notion of compactness helps us distinguish $(0,1) \cong \mathbb{R}$ from $[0,1] \subseteq \mathbb{R}$ and connectedness helps distinguish $(-1,1)$ from $(-1,0) \cup (0,1)$. Other point-set motivated invariants include seperation axioms, countability conditions, and specializations of the above (e.g path connectedness, Lindeloff, etc). However, neither of these invariants help immediately distinguish between $[0,1]$ and S^1. We use this to motivate the invariant of $\pi_1(X)$ the fundamental group of X. Since the focus of this paper is on the algebra, we will be referring to [AH] for the proofs of the results.

1.1. **Paths and Path-homotopies.** Let X be a topological space. Recall that $\gamma : [0,1] \to X$ continuous is a path in X from $x_0 := \gamma(0) \to x_1 := \gamma(1)$. We make the following definition

Definition 1.1. Two paths $\gamma_1, \gamma_2 : [0,1] \to X$ with $\gamma_1(0) = \gamma_2(0) = x_0$ and $\gamma_1(1) = \gamma_2(1) = x_1$ are called path homotopic if there exists some $\Gamma : [0,1] \times [0,1] \to X$ continuous satisfying $\Gamma(0,t) = \gamma_0(t)$, $\Gamma(1,t) = \gamma_1(t)$ for all $t \in [0,1]$, and $\Gamma(s,0) = x_0$ and $\Gamma(s,1) = x_1$ for all $s \in [0,1]$. Γ is said to be the path homotopy from $\gamma_0 \to \gamma_1$

Choosing $x_0, x_1 \in X$ it is easy to verify that being path homotopic defines an equivalence relation on the set of paths from $x_0 \to x_1$ in X.

1.2. **First Fundamental Group.** Now, let us consider the special setting of above where $x_0 = x_1$- loops centered at x_0.

Definition 1.2. Let X be a topological space and $x_0 \in X$ a distinguished point

$$\pi_1(X, x_0) = \{\text{loops in X at } x_0\}/\{\text{path homotopy}\}$$

Lemma 1.3. *Given X, x_0 as above, then $\pi_1(X, x_0)$ has a group law given by concatenation of loops. For a loop $\alpha : [0,1] \to X$ a loop, the inverse of $[\alpha] \in \pi_1(X, x_0)$ is given by $[\overline{\alpha}]$ where $\overline{\alpha}$ is the reversed parameterization of α.*

Lemma 1.4. *Given X a topological space and $x_0, x_1 \in X$ with $\eta : [0,1] \to X$ a path from $x_0 \to x_1$ we have that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ via the map $[\gamma] \mapsto [\overline{\eta} \cdot \gamma \cdot \eta]$ where $\cdot$ indicates concatenation of paths.*

Lemma 1.5. *Let X, Y be topological spaces and $x_0 \in X$ and $y_0 \in Y$ with $f : X \to Y$ continuous mapping $x_0 \mapsto y_0$ then f induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\alpha] \mapsto [f \circ \alpha]$. If f is a homeomorphism then this map is an isomorphism.*

Definition 1.6. A space X is semi-locally simply connected provided that for all points $x \in X$ there is a neighborhood $x \in U \subseteq X$ so that the map $i_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion is the trivial map on groups.

Proof. Proofs of Lemma 1.3 - Lemma 1.5 are elementary but involve some detailed checking and can be found in [AH] □

Remark 1.7. A corollary of Lemma 1.3 is that, for a path connected space X and a a point $x_0 \in X$, $\pi_1(X, x_0)$ is independent of x_0. Thus, we define $\pi_1(X) := \pi_1(X, x_0)$ and call this the first fundamental group of X. Lemma 1.5 shows that this is invariant under homeomophism.

Example 1.8. We can show that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1([0,1]) \cong \{0\}$ distinguishing these spaces!

1.3. **Covering Space Theory.** It is often quite difficult to compute $\pi_1(X)$ (perhaps highlighted by the lack of detail in the example above). So, we look at the following when X is connected

Definition 1.9. Let X be given. We have that $p : \widetilde{X} \to X$ is a covering space if $\forall x \in X$ there is some neighborhood $x \in U$ so that $p^{-1}(U) = \bigsqcup_{i \in I} \widetilde{U}_i$ (I nonempty) with $\widetilde{U}_i \subseteq \widetilde{X}$ open and $p|_{\widetilde{U_i}} : \widetilde{U}_i \to U_i$ is a homeomorphism. Note that this is surjective by the non-emptiness of I. We say that a covering space is finite if the preimage of any point is finitely many points. It can be shown that the number of preimages is independent of the chosen points and, thus, a property of the covering space itself. We call this finite value the index of the cover.

Lemma 1.10. *If $p : \widetilde{X} \to X$ is a covering space mapping $\widetilde{x_0} \mapsto x_0$ then $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ is an injection. So, $\pi_1(\widetilde{X}, \widetilde{x_0}) \subseteq \pi_1(X, x_0)$ a subgroup.*

Definition 1.11. Let $p_1 : \widetilde{X}_1 \to X$, $p_2 : \widetilde{X}_2 \to X$ be convering spaces. A morphism of covering spaces is a continuous map $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_1 = p_2 \circ f$. If f is a homeomorphism then this is an isomorphism of covering spaces. We denote $\mathrm{Aut}(\widetilde{X_1}/X)$ as the group covering space isomorphisms of $X_1 \to X_1$. We note that this acts on the fibres of the points of X due to the commutativity required. This is called the group of covering space/deck transformations.

Definition 1.12. We say that a cover $p : \widetilde{X} \to X$ is Galois if $\mathrm{Aut}(\widetilde{X}/X)$ acts transitively on the fibres of X.

Lemma 1.13. *A cover $p : \widetilde{X} \to X$ is Galois iff $\pi_1(\widetilde{X}, \widetilde{x_0}) \subseteq \pi_1(X, x_0)$ is a normal subgroup. A corollary of this is that a universal cover (i.e. a cover with trivial fundamental group) is Galois.*

Lemma 1.14. *Let $p : \widetilde{X} \to X$ be a covering space. Let $x_0 \in X$ be given and some $\widetilde{x_0} \in p^{-1}(x_0)$ in the fibre be chosen. Then if γ is a loop at x_0 then there is a unique lift of γ to $\widetilde{X}$, call it $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ so that $\widetilde{\gamma} = \widetilde{x_0}$ and note that, being a lift, $\widetilde{\gamma}(1) \in p^{-1}(x_0)$. This gives a group action on...*

Theorem 1.15. *If X is a path connected, semilocally simply connected space. Let $x_0 \in X$ be given then we have a bijection (motivated by what we see in Lemma 1.10)*

$$\begin{aligned}\frac{\{p : \widetilde{X} \to X \ \textit{p.c covering space}\}}{\{\cong \ \textit{of covering spaces}\}} &\leftrightarrow \{\textit{conjugacy classes of subgroups of } \pi_1(X, x_0)\} \\ &\leftrightarrow \{\pi_1(X, x_0)\textit{-sets}\}\end{aligned}$$

Furthermore, we have that Galois covers correspond to conjugacy classes of normal subgroups $N \subseteq \pi_1(X, x_0)$. These Galois covers then have that the group of deck transformations is $\frac{\pi_1(X,x_0)}{N}$. This looks quite similar to main theorem of Galois Theory!

Proof. Proofs of all the results above are found in [AH] ☐

2. Algebraic Buildup

Our goal now is to extend the idea of covering spaces to rings and beyond to address the initial problem of understanding the topology of $\mathrm{Spec}(R)$ from R itself (again for a specific class of rings R). To do so, we must first build some definitions and ideas. Let us start with the idea of an affine algebraic varieties and some associated objects and maps [SKKT]

2.1. **Affine Algebraic Varieties and Coordinate Rings.**

Definition 2.1. Let K be an algebraically closed field (this will be an assumption henceforth), an affine algebraic variety is the common zero locus of a collection $\{F_i\}_{i\in I} \subseteq K[x_1,\ldots,x_n]$ of polynomials in K^n. We call $V = V(\{F_i\}_{i\in I}) \subseteq K^n$. We note that this just depends on the ideal generated by $\{F_i\}_{i\in I}$ in $K[x_1,\ldots,x_n]$

Lemma 2.2. *We can define a topology on K^n where the open sets are given by $K^n \backslash V(\{f_i\}_{i\in I})$ for polynomials $f_i \in K[x_1,\ldots,x_n]$*

Proof. First we note that K^n is closed since it is $V(0) = K^n$ where 0 denotes the zero polynomial. Similarly we observe that $V(c) = \varnothing$ where $c \neq 0$ is a constant polynomial (nowhere vanishing).

Now, let us show that the closed sets are closed under finite unions (it suffices to do it for two and then inducting). Suppose that
$V(\{F_{i,1}\}_{i\in I_1}), V(\{F_{i,2}\}_{i\in I_2})$ are a collection of closed sets. We observe that

$$V(\{F_{i,1}\}_{i\in I_1}) \cup V(\{F_{i,2}\}_{i\in I_2}) = V\left(\{F_{i,1}\}_{i\in I_1}\{F_{i,2}\}_{i\in I_2}\right)$$

Where the right hand-side indicates the elements of the form f, g where $f \in \{F_{i,1}\}_{i\in I_1}$ and $g \in \{F_{i,2}\}_{i\in I_2}$. This isn't too hard to show we consider some $x \in V(\{F_{i,1}\}_{i\in I_1}) \cup V(\{F_{i,2}\}_{i\in I_2})$. We have that there is some $f' \in \{F_{i,1}\}_{i\in I_1}$ or $f' \in \{F_{i,2}\}_{i\in I_2}$ so that $f(x) = 0$. WLOG consider the first case, then for any $g \in \{F_{i,2}\}_{i\in I_2}$ we have that $fg \in \{F_{i,1}\}_{i\in I_1}\{F_{i,2}\}_{i\in I_2}$ and $fg(x) = f(x)g(x) = 0$ so $x \in V\left(\{F_{i,1}\}_{i\in I_1}\{F_{i,2}\}_{i\in I_2}\right)$. The other direction is similar and follows from the fact that K is a domain.

Finally, we show that closed sets are closed under arbitrary intersections. This is a similar argument to the above where, letting $\{V(\{F_{i,j}\}_{i\in I})\}_{j\in J}$ be a collection of closed sets. Then we have that

$$\bigcap_{j\in J} V(\{F_{i,j}\}_{i\in I}) = V\left(\bigcup_{j\in J}\{F_{i,j}\}_{i\in I}\right)$$

∎

Definition 2.3. The above gives us a topology on K^n called the Zariski Topology. This set equipped with the topology is denoted $\mathbb{A}^n_K$ affine n-space.

Definition 2.4. Morphism of affine algebraic varieties. Suppose that $X \subseteq \mathbb{A}^n_K$ and $Y \subseteq \mathbb{A}^m_K$ are affine algebraic varieties. A map $f : X \to Y$ is said to be a morphism of affine algebraic varieties provided that f is a restriction of a map of the following form

$$\begin{aligned} p : \mathbb{A}^n_K &\to \mathbb{A}^m_K \\ (x_1, \dots, x_n) &\mapsto (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n)) \end{aligned}$$

Where $F_i \in K[x_1, \dots, x_n]$ is a polynomial. Thus, f is a coordinate-wise polynomial map from $X \to Y$. We further say that this map is an isomorphism if it is bijective with an inverse that is also a morphism of affne algebraic varieties.

Lemma 2.5. *Suppose that $f : X \to Y$ is a morphism of affine algebraic varieties. Then we have that f is continuous with respect to the Zarinski Topology.*

Proof. We note that $f : X \to Y$ is a restriction of a polynomial map as above $p : \mathbb{A}^n_K \to \mathbb{A}^m_k$. It suffices to show that this is continous since any restriction will be continuous with respect to the subspace topology which is what we endow X, Y resp. with.

So consider such a polynomial $p : \mathbb{A}^n_K \to \mathbb{A}^m_k$ given by the following,

$$\begin{aligned} p : \mathbb{A}^n_K &\to \mathbb{A}^m_K \\ (x_1, \dots, x_n) &\mapsto (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n)) \end{aligned}$$

It suffices to show that inverse images of closed sets are closed (elementary point-set topology result). So consider $V(\{G_i\}_{i \in I})$ this is exactly $V(\{G_i \circ p\}_{i \in I})$ which is closed in $\mathbb{A}^m_K$. □

Definition 2.6. Let $V \subseteq \mathbb{A}^n_K$ an affine variety be given. We note that given any polynomial $p \in K[x_1, \dots, x_n]$ the map $p|_V : V \to K$. So, we define the coordinate ring of V to be $K[x_1, \dots, x_n]|_V$ and denote this $C(V)$. Noting that $K[x_1, \dots, x_n] \twoheadrightarrow C(V)$ gives a surjective with kernel $I(V)$, the ideal consisting of polynomials that vanish on the entirety V. Therefore, we have that $C(V) \cong K[x_1, \dots, x_n]/I(V)$- a finitely generated K algebra!

Lemma 2.7. *Suppose that we have a map of varieties $f : V \to W$ where $V \subseteq \mathbb{A}^n_K$ and $W \subseteq \mathbb{A}^m_K$ then we get an induced K-algebra homomorphism,*

$$\begin{aligned} f^* : C(W) &\to C(V) \\ p &\mapsto p \circ f \end{aligned}$$

Proof. §2.4 [SKKT] □

Lemma 2.8. *Suppose that $V \subseteq \mathbb{A}^n_k$ is an affine algebraic variety with coordinate ring $C(V)$. We have that $C(V)$ is a reduced ring too (i.e. it has no non-zero nilpotents). So we see that it is a reduced, finitely-generated K-algebra*

Proof. Let $V = V(\{F_i\}_{i \in I}) \subseteq \mathbb{A}^n_K$ be an affine algebraic variety. Then the coordinate ring is $C(V) \cong K[x_1, \dots, x_n]/I(V)$. We recall that $I(V)$ is a radical ideal (in particular it is the radical of $(\{F_i\}_{i \in I})$). Then we show the general claim that if I is a radical ideal then $K[x_1, \dots, x_n]/I$ has no nontrivial nilpotents. Suppose that $f^n = 0$ in $K[x_1, \dots, x_n]/I$. Therefore, we have that $f^n \in I \implies f \in \mathrm{rad}(I) = I \implies f = 0$ in the quotient ring. Thus, we have that $K[x_1, \dots, x_n]/I$ has no nontrivial nilpotents! □

Lemma 2.9. *Suppose that $\mathcal{C}$ is the category of affine algebraic varieties over K with morphisms given by maps of varieties. Suppose that $\mathcal{D}$ is the category of reduced, finitely generated K algebras. There is a contravariant equivalence between these categories (modulo isomorphism on either side).*

Proof. Above we see that $V \to C(V)$ gives a map from $\mathcal{C} \to \mathcal{D}$ where the map on morphisms is given by the pullback described in Lemma 2.7.

We defer the details of the inverse map of this result to §2.5 [SKKT] which shows that every reduced, finitely generated K-algebra is isomorphic to the coordinate ring of some affine algebraic variety over K. □

2.2. **Etalé Maps of Rings.** [DC]

Now we consider the setup above with $K = \mathbb{C}$. We note that $\mathbb{C}^n$ already possess a topology (not true for arbitrary such K). Let an affine algebraic variety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ be given. We may consider $M(X)$ denote considering $X \subseteq \mathbb{C}^n$ with the usual topology. Furthermore, we noted that given another affine algebraic variety $Y \subseteq \mathbb{A}^n_{\mathbb{C}}$ and $M(Y) \subseteq \mathbb{C}^n$ that morphisms of affine algebraic varities $f : M(Y) \to M(X)$ (which are continuous since they are restrictions of coordinate-wise polyomials (they are also continuous in the Zariski topology which is coarser than the usual topology)) are in bijection with $\mathbb{C}$-algebra homomorphism from $f^* : C(X) \to C(Y)$. We aim to understand the criterion on f^* required for the underlying f to be a covering space (since, we saw earlier that understanding covering spaces is intrinsic to realizing $\pi_1(M(X))$). Then, once we have this, the equivalence of categories will allow us to define a fundamental group on the underlying coordinate rings (which can be any reduced, finitely generated K-algebra as seen as above)- a purely algebraic construction! However, the Zariski topology tends to be quite coarse so we use this as intuitional inspiration and explore the following criterion for such a map and then generalize it to define the etalé fundamental group in the next section and then bring it back home

Definition 2.10. Unramified map of rings. Let $f : R \to S$ be a map of rings. We say that this is an unramified map provided that, for all prime ideal $\mathfrak{p} \subseteq S$ we have that induced map on local rings $\widetilde{f} : R_{f^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$ preserves the maximal ideal in the following sense- $\widetilde{f}$ maps $f^{-1}(\mathfrak{p})\ R \to \mathfrak{p}\ S$.

Definition 2.11. Flat map. Let $f : R \to S$ be a map of rings. The map f is said to be flat if S is flat as an R-module (in the usual sense).

Definition 2.12. Map of finite type. Let $f : R \to S$ be a map of rings. The map f is said to be of finite presentation if S is generated presented as an R-module.

Definition 2.13. Finite Etalé Map of Rings. We say that $f : R \to S$ is an etalé map of rings provided that it is unramified, flat and of finite type!

Definition 2.14. Suppose that R, S are rings with spectra $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ respectively. We recall that a map $f : R \to S$ induces a map $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ via contraction. If the initial map is a Finite etalé map then we say the the latter is a covering map

Definition 2.15. Suppose that R, S are rings with spectra Spec(R) and Spec(S) respectively. In the above setting with $f : R \to S$ Finite etalé, we say that a ring homomorphism $g : S \to S$ is a covering transformation (also a deck transformation) if $g|_{f(R)}$ is the identity. This is precisely the automorphisms of S considering it as an R-algebra via f! We will denote this by $\mathrm{Aut}_f(S/R)$

3. Etalé Fundamental Group

We now use the previous build up to define, for rings A, the etalé fundamental group of its spectra- Spec(A). Just as in the topological case where we restricted ourselves to connected topological spaces, we will consider the case when Spec(A) is connected. This precisely corresponds to the ring theoretic criterion of A possessing no nontrivial idempotent(s). Thus we see that this restricts to the case when A cannot be decomposed as the direct sum of nontrivial rings (existence of an idempotent $e \in A$ gives a decomposition $A \cong eA \oplus (1-e)A$).

If we recall the "Galois Correspondence" from section 1.3, we want such a group, call it $\pi_1^{\text{ét}}(\text{Spec } A)$, to satisfy the following:

> If we have some $N \subseteq \pi_1^{\text{ét}}(\text{Spec } A)$ a normal subgroup then there is a corresponding covering map $\tilde{f} : \text{Spec}(B) \to \text{Spec}(A)$ so that $\mathrm{Aut}_f(B/A) \cong \frac{\pi_1^{\text{ét}}(\text{Spec } A)}{N}$!

Definition 3.1. Etalé Fundamental Group. Suppose that A is a connected ring (i.e. has connected spectrum) then we define

$$\pi_1^{\text{ét}}(\text{Spec } A) = \varprojlim \mathrm{Aut}_f(B/A)$$

Where we let B, f range over finite etalé maps $f : A \to B$ with connected B. This ends up giving us a functor between the appropriate categories [DC]!

Lemma 3.2. *Let A be a ring and consider $\pi_1^{ét}(Spec\ A)$ as defined above. This satisfies the wishlist item: If we have some $N \subseteq \pi_1^{ét}(Spec\ A)$ a normal, open (in the profinite topology) subgroup then there is a corresponding covering map $\tilde{f} : Spec(B) \to Spec(A)$ so that $Aut_f(B/A) \cong \frac{\pi_1^{ét}(Spec\ A)}{N}$*

Proof. This is immediate from its definition as the inverse limit of such required groups! □

3.1. Profinite Completions of the Ordinary Fundamental Group.

Definition 3.3. Suppose that G is a group. Then we recall that the profinite completion of G is

$$\widehat{G} = \varprojlim G/N$$

Where N ranges over all the normal subgroups of G with finite index!

Theorem 3.4. *Now let us revert to the case where we are looking at affine algebraic varieties over $\mathbb{C}$. Let X be such and let $C(X)$ be its coordinate ring and $M(X)$ considering $X \subseteq \mathbb{C}^n$ as a subspace with the usual topology.*

Then we have that,

$$\pi_1^{ét}\left(Spec\ C(X)\right) \cong \pi_1(\widehat{M(X)})$$

Proof. This result arises by work of Grothendieck. This uses the Riemann Existence Theorem [Har] which shows that any finite covering space of $M(X)$ is given by a polynomial map of $M(Y) \to M(X)$ with covering transformations being maps of varieties! This gives that there is a bijection between finite covering spaces of $M(X)$ and finite étale maps of the form $C(X) \to B$ some variety. Put simply there is a correspondence between the topological notion of covering spaces of $M(X)$ and (finite etale) coverings of $C(X)$ as defined in Definition 2.15. This correspondence leads to the above result [Gro] □

3.2. **The Field Case.** Now we note the following result. We don't provide a detailed proof here since it isn't quite within the scope of the rest of the paper, but, it is an extremely interesting result and worth discussing.

Lemma 3.5. *Suppose that K is a field. Then we have that the finite etalé morphisms of connected rings $f : K \to B$ is exact the finite seperable field extensions $K \hookrightarrow L$*

Proof. [SP] □

Theorem 3.6. *Suppose that K is a field then*

$$\pi_1^{ét}\left(Spec\ K\right) \cong Gal(\overline{K}/K)$$

Where $\overline{K}$ is the separable, algebraic closure of K. This is the absolute Galois Group of K.

Proof. This follows from the previous lemma noting that $\pi_1^{\text{ét}}\left(\text{Spec } K\right)$ is simply the inverse limit of finite, separable field extensions of K. □

4. Dessin d'Enfants

A Dessin d'Enfant is a finite covering of $\mathbb{C} \setminus \{0, 1\}$ (which we will do by a Riemann Surface which arises from most $M(A)$ where A is a affine algebraic variety). In this section we will build up motivation of studying such objects and also look at how to find and represent such coverings.

We let $S = \{0, 1\}$ indicate the two point set we wish to disclude. We observe that the coordinate ring of $\mathbb{C}\setminus S$ is given by $C(\mathbb{C}\setminus S) = \mathbb{C}\left[x, \frac{1}{x(x-1)}\right]$ the localization at $p(x) = x(x-1)$. Theorem 3.4 gives us that $\pi_1^{\text{ét}}\left(\text{Spec } C(\mathbb{C} \setminus S)\right) = \pi_1(\widehat{M(\mathbb{C} \setminus S)})$.

Now, we note the following. We have that $\mathbb{Q} \hookrightarrow \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right]$ and we also have the (homomorphism) map $\mathbb{Q}\left[x, \frac{1}{x(x-1)}\right] \to \mathbb{C}\left[x, \frac{1}{x(x-1)}\right]$ as above. Then the get the maps $\text{Spec } \mathbb{C}\left[x, \frac{1}{x(x-1)}\right] \to \text{Spec } \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right] \to \text{Spec } \mathbb{Q}$. Then, using the functoriality of taking the etalé fundamental group, we get the following sequence of groups

$$\pi_1^{\text{ét}}\left(\text{Spec } \mathbb{C}\left[x, \frac{1}{x(x-1)}\right]\right) \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right]\right) \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\right)$$

This sequence turns out to be exact [DC] giving rise to the following short exact sequence,

$$0 \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{C}\left[x, \frac{1}{x(x-1)}\right]\right) \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right]\right) \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\right) \to 0$$

This precisely gives rise to, using out work from Section 3,

$$0 \to \pi_1(\widehat{M(\mathbb{C} \setminus S)}) \to \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right]\right) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 0$$

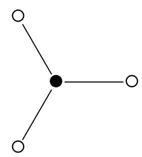
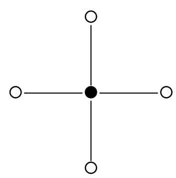
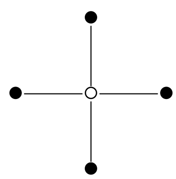
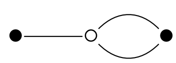
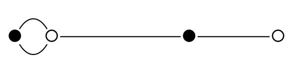
Thus, we observe that understanding the coverings of $\mathbb{C} \setminus S$ helps us understand $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ an object of great important in algebraic number theory. Note that there was nothing special about out particular choice of $\mathbb{C} \setminus S$ as an affine algebraic scheme- we could have used most reasonable affine algebraic schemes over $\mathbb{Q}$ to begin. This is known as the fundamental exact sequence.

Now, we note a consequence of this. First let $X = \pi_1^{\text{ét}}\left(\text{Spec } \mathbb{Q}\left[x, \frac{1}{x(x-1)}\right]\right)$. We observe, from the fundamental exact sequence, that $\pi_1(\widehat{M(\mathbb{C} \setminus S)})$ is a normal subgroup of $\pi_1^{\text{ét}}(\text{Spec } X)$ (as $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) is a quotient of them. Then we see that the action of $\pi_1^{\text{ét}}(\text{Spec } A)$ on itself via conjugation descents to an action on $\pi_1(\widehat{M(\mathbb{C} \setminus S)})$. Thus, we have a homomorphism $\pi_1^{\text{ét}}(\text{Spec } A) \to \text{Aut}(\pi_1(\widehat{M(\mathbb{C} \setminus S)}))$. Now we recall that the group of outer isomorphisms is given by the quotient of the automorphism group by the inner automorphisms of the group (acting on oneself via conjugation) $\text{Aut}(\pi_1(\widehat{M(\mathbb{C} \setminus S)})) \to \text{Out}(\pi_1(\widehat{M(\mathbb{C} \setminus S)}))$; so this map has kernel of the inner automorphisms. Now, we note that elements of $\pi_1(\widehat{M(\mathbb{C} \setminus S)}) \subseteq \pi_1^{\text{ét}}(\text{Spec } A)$ map to the identity of the outer automorphism group. Thus combining our prior maps, we get an induced map $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Out}(\pi_1(\widehat{M(\mathbb{C} \setminus S)}))$. So, we get that the Absolute Galois group permutes covers of $M(\mathbb{C} \setminus S)$. We finally note that, via topology, $\mathbb{C} \setminus S$ is homotopy equivalent to $S^1 \vee S^1$ (via a deformation retraction) which gives that $\pi_1(M(\mathbb{C} \setminus S)) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$. Thus, we have the following map $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Out}(\widehat{\mathbb{Z} * \mathbb{Z}})$. This is why it is a good idea to study what these covers are. This is further highlighted by the following theorem of Belyî

Theorem 4.1 (Belyî)**.** *Every Algebraic Curve defined over $\overline{\mathbb{Q}}$ can be represented as a covering of $\mathbb{C}$ by a compact Riemann Surface that is ramified at, at most, two points (i.e. a covering of $\mathbb{C}\setminus\{0,1\}$ via precomposition). A presented corollary is that the map $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Out(\widehat{\mathbb{Z} * \mathbb{Z}})$ is injective as the given outer automorphism is faithful*

Proof. Look at [Bey] using the notation $\mathbb{P} \setminus \{0, 1, \infty\}$ □

4.1. **Examples of Dessins.** We conclude this paper by writing out some examples of such covers. We note that, in the language of Complex Analysis, such a map of affine varieties (compact Riemann surfaces) (potentially) ramified at 0 and 1. We will take the Riemann surface to be the Riemann Sphere, $\hat{\mathbb{C}}$, and so the above corresponds to rational maps which are potentially ramified at $\{0,1\}$ (equivalently maps from $\hat{\mathbb{C}}$ to itself ramified at $0,1,\infty$). So we can consider maps such as $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $x \mapsto x^3$ or $x \mapsto 1-x^4$ and such. Given such a map, we can construct drawings indicating the preimages of $1,0$ given by the white and black nodes respectively. Then we have that the edges are the preimage of the interval $[0,1] \subseteq \mathbb{C}$. Below we have examples of such maps and their associated *Dessins* [Zap]

Dessin Graph	Map from $\hat{\mathbb{C}}$
	x^3
	x^4
	$1-x^4$
	$\frac{(x-2)(1-x)^2}{x}$
	$\frac{(x^2-6x-3)^2}{64x}$
	$8x^4-8x^2+1$

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